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Construction and Properties of Multiwavelet Packets with Arbitrary Scale and the Related Algorithms of Decomposition and Reconstruction

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Abstract—A more flexible method of constructing multiwavelet packets with the scale $a \geq 2$ from the same set of multiwavelets is discussed in this paper. The properties of these packets are studied. The formulae for performing iterations and decomposition are given. $L^2(R)$ can be further decomposed by these multiwavelet packets and good bases of $L^2(R)$ are given. Finally, we provide the algorithms for the decomposition and reconstruction using these multiwavelet packets. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Multiwavelets, Multiwavelet packets, Scale, Scaling function vector, Matrix sequence.

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1. INTRODUCTION

The symmetry of wavelet basis is important in signal compression. But the continuous orthogonal scalar wavelet bases with compact support do not have symmetry (see [1]). Multiwavelet, initiated by Goodman *et al.* [2], overcomes the drawback. Since then multiwavelet has received considerable attention from the wavelet research communities both in theory and in applications. For multipass filter and more flexible wavelets, Geronimo *et al.* [3] introduced the theory of multiwavelet with scale a leading to encouraging results. Marasovich composed and studied biorthogonal wavelet [4].

Wavelet packets were introduced by Coifman *et al.* [5,6] aiming at better frequency field and thereby providing a more efficient decomposition of signals containing both transient and stationary components. The advantages of wavelet packets and their promising features in various applications have attracted a lot of interest and effort in recent years. Readers are referred to some of these references [7–14]. In addition Coifman *et al.* [6] introduced biorthogonal wavelet packets using splines, Cheng [10] introduced packets with matrices, Yang *et al.* [11] provided a method of constructing orthogonal multiwavelet packets, and Leng *et al.* [12,14] discussed the biorthogonal case and its properties.

The construction of multiwavelet packets with scale $a \geq 2$, where $a \in Z$, is harder than the construction of multiwavelet packets with the same scale because of the difference between the multiplicity of multiwavelet with that particular scale and the multiplicity of its scaling function vector. We propose, in Section 2, a flexible method of constructing multiwavelet packets by applying the same multiwavelets. In Section 3, the properties, including biorthogonality and formulae for performing iterations and decomposition, of multiwavelet packets with the scale $a \geq 2$ are examined. In Section 4, a study of the decomposition of $L^2(R)$ with these packets and two new bases of $L^2(R)$ are given, which may be used to observe signals with high frequency. In Section 5, we propose an algorithm for the decomposition and reconstruction using these packets by applying the results in Section 4. A note on the derivation of the related results for orthogonal multiwavelet packets with the scale $a \geq 2$ by using the biorthogonal case is included. In this paper, we adopt the following symbols. Let $f_l(x) \in L^2(R)$ be a function vector with elements

$$f_{l;j,k}(x) = a^{j/2} f_l \left(a^j x - k\right),$$

where $l = 1, 2, \ldots, n$, $j, k \in Z$. The Fourier transform of a function vector is defined as the Fourier transform of all of its elements. The inner product of two function vectors $f(x) = (f_1, f_2, \ldots, f_n)^\top$ and $g(x) = (g_1, g_2, \ldots, g_n)^\top$ is defined as the matrix

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x) g(x)^* \, dx,$$

where the superscript "*" denotes the conjugate transpose of a matrix.

2. BIORTHOGONAL MULTIWAVELET PACKETS WITH ARBITRARY SCALE

This section provides some basic definitions followed with the construction of biorthogonal multiwavelet packets with the scale a.

A vector $\Phi(x) = (\varphi_1, \varphi_2, \ldots, \varphi_t)^\top$ is called a scaling function vector with scale a, if it can produce a multiresolution analysis (MRA) $\{V_j\}_{j\in Z}$ and satisfies

$$\Phi(x) = \sum_{k\in Z} P_k \Phi(ax - k), \tag{2.1}$$

where $\{P_k\}_{k\in Z}$ is a matrix sequence, consisting of $t \times t$ matrices, of Φ. The Fourier transform of (2.1) is given by

$$\hat{\Phi}(\omega) = P(z)\hat{\Phi}\left(\frac{\omega}{a}\right), \qquad z = e^{-i\omega/a}, \tag{2.2}$$

where $P(z) = (1/a)\sum_{k\in Z} P_k z^k$ is known as the matrix symbol of Φ.

The subspace V_j is given by

$$V_j = \mathrm{clos}_{L^2(R)}\left\langle \varphi_{l;j,k} : 1 \le l \le t,\ k \in Z \right\rangle, \qquad j \in Z, \tag{2.3}$$

where "clos" denotes the closure of a space or a space produced by a function. $\{V_j\}_{j\in Z}$ is a multiresolution analysis generated by Φ if it satisfies

(1) $\cdots \subset V_0 \subset V_1 \subset V_2 \subset \cdots$;
(2) $\mathrm{clos}_{L^2(R)}\left\langle \bigcup_{j\in Z} V_j \right\rangle = L^2(R)$;
(3) $\bigcap_{j\in Z} V_j = \{0\}$;
(4) $f(x) \in V_j \Leftrightarrow f(ax) \in V_{j+1},\ j \in Z$;
(5) $\{\varphi_{l;j,k} : 1 \le l \le t,\ k \in Z\}$ is a Riesz basis of V_j.

A function vector $\Psi(x) = (\psi_1, \psi_2, \ldots, \psi_{(a-1)t})^{\top}$ is called a multiwavelet with scale a associated with Φ if $\{\psi_{l;j,k} : 1 \le l \le (a-1)t,\ k \in Z\}$ is a Riesz basis of W_j, where $V_{j+1} = V_j \oplus W_j$, $j \in Z$. Then there exists a matrix sequence of Ψ $\{Q_k\}_{k\in Z}$, in which each element of the sequence is a matrix of order $(a-1)t \times t$, such that

$$\Psi(x) = \sum_{k\in Z} Q_k \Phi(ax - k). \tag{2.4}$$

The Fourier transform of (2.4) is given by

$$\hat{\Psi}(\omega) = Q(z)\hat{\Phi}\left(\frac{\omega}{a}\right), \tag{2.5}$$

where $Q(z) = (1/a)\sum_{k\in Z} Q_k z^k$ is known as the matrix symbol of Ψ.

If Ψ is multiwavelet associated with the scaling function vector Φ, then there exist function vectors $\tilde{\Phi}(x) = (\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_t)^{\top}$ and $\tilde{\Psi}(x) = (\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_{(a-1)t})^{\top}$, which are called dual scaling function vector of Φ and dual multiwavelet of Ψ, respectively, if they satisfy

$$\left\langle \Phi(x), \tilde{\Phi}(x-n) \right\rangle = \delta_{0,n} I_t,$$

$$\left\langle \Phi(x), \tilde{\Psi}(x-n) \right\rangle = \left\langle \tilde{\Phi}(x-n), \Psi(x) \right\rangle = 0,$$

and

$$\left\langle \Psi(x), \tilde{\Psi}(x-n) \right\rangle = \delta_{0,n} I_{(a-1)t}, \qquad n \in Z,$$

where

$$\delta_{0,n} = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and I_n denotes the unit square matrix of order n for $n = 1, 2, \ldots$. Here Φ and $\tilde{\Phi}$ are biorthogonal scaling vectors, Ψ and $\tilde{\Psi}$ are biorthogonal multiwavelets. There exists matrix sequences $\{\tilde{P}_k\}_{k\in Z}$ and $\{\tilde{Q}_k\}_{k\in Z}$, the former one consists of matrices of order $t \times t$ and the latter consists of matrices of order $(a-1)t \times t$, such that

$$\tilde{\Phi}(x) = \sum_{k\in Z} \tilde{P}_k \tilde{\Phi}(ax - k), \qquad \tilde{\Psi}(x) = \sum_{k\in Z} \tilde{Q}_k \tilde{\Phi}(ax - k). \tag{2.6}$$

The Fourier transforms of (2.6) are given by

$$\hat{\tilde{\Phi}}(\omega) = \tilde{P}(z)\hat{\tilde{\Phi}}\left(\frac{\omega}{a}\right), \tag{2.7}$$

$$\hat{\tilde{\Psi}}(\omega) = \tilde{Q}(z)\hat{\tilde{\Phi}}\left(\frac{\omega}{a}\right), \tag{2.8}$$

where $\tilde{P}(z) = (1/a)\sum_{k\in Z}\tilde{P}_k z^k$ and $\tilde{Q}(z) = (1/a)\sum_{k\in Z}\tilde{Q}_k z^k$ are the matrix symbols of $\tilde{\Phi}$ and $\tilde{\Psi}$, respectively. $\{\psi_{l;j,k} : 1 \le l \le (a-1)t, j,\ k \in Z\}$ and $\{\tilde{\psi}_{l;j,k} : 1 \le l \le (a-1)t,\ j,k \in Z\}$ are biorthogonal wavelet bases of $L^2(R)$. Suppose

$$\tilde{V}_j = \operatorname{clos}_{L^2(R)}\langle\tilde{\varphi}_{l;j,k} : 1 \le l \le t,\ k \in Z\rangle, \qquad j \in Z, \tag{2.9}$$

$$\tilde{W}_j = \operatorname{clos}_{L^2(R)}\left\langle\tilde{\psi}_{l;j,k} : 1 \le l \le (a-1)t,\ k \in Z\right\rangle, \qquad j \in Z, \tag{2.10}$$

are dual spaces of V_j and W_j, respectively, and satisfy

$$\tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j, \quad V_j \perp \tilde{V}_j, \quad W_j \perp \tilde{W}_j, \qquad j \in Z.$$

In order to construct biorthogonal multiwavelet packets with the scale a, we divide Ψ into $a-1$ function vectors each of t-dimension, arbitrarily as follows.

$$\Psi_i(x) = (\psi_{i_1}, \psi_{i_2}, \dots, \psi_{i_t})^\top, \qquad i = 1, 2, \dots, a-1. \tag{2.11}$$

Similarly for $\tilde{\Psi}(x)$, we have

$$\tilde{\Psi}_i(x) = \left(\tilde{\psi}_{i_1}, \tilde{\psi}_{i_2}, \dots, \tilde{\psi}_{i_t}\right)^\top, \qquad i = 1, 2, \dots, a-1. \tag{2.12}$$

Each of the matrices Q_k and $\tilde{Q}_k$ is divided into $(a-1)$ matrices corresponding to Ψ and $\tilde{\Psi}$ as follows.

$$Q_k = \left(Q_k^{(1)\top}, Q_k^{(2)\top}, \dots, Q_k^{(a-1)\top}\right)^\top, \qquad \tilde{Q}_k = \left(\tilde{Q}_k^{(1)\top}, \tilde{Q}_k^{(2)\top}, \dots, \tilde{Q}_k^{(a-1)\top}\right)^\top. \tag{2.13}$$

Suppose

$$P_k^{(0)} = P_k,\ \tilde{P}_k^{(0)} = \tilde{P}_k, \qquad P_k^{(i)} = Q_k^{(i)},\ \tilde{P}_k^{(i)} = \tilde{Q}_k^{(i)}, \quad i = 1, 2, \dots, a-1, \quad k \in Z, \tag{2.14}$$

$$P^{(i)}(z) = \frac{1}{a}\sum_{k\in Z} P_k^{(i)} z^k, \qquad \tilde{P}^{(i)}(z) = \frac{1}{a}\sum_{k\in Z}\tilde{P}_k^{(i)} z^k, \qquad i = 0, 1, \dots, a-1, \tag{2.15}$$

and that $\Psi_0 = \Phi$ and $\tilde{\Psi}_0 = \tilde{\Phi}$. The biorthogonal multiwavelet packets with scale a are given below.

DEFINITION 2.1. *The function vector collections* $\{\Psi_{al+i} : l = 0, 1, \dots,\ i = 0, 1, \dots, a-1\}$ *and* $\{\tilde{\Psi}_{al+i} : l = 0, 1, \dots,\ i = 0, 1, \dots, a-1\}$ *are called the biorthogonal multiwavelet packets of* Φ *and* $\tilde{\Phi}$, *respectively, where*

$$\Psi_{al+i}(x) = \sum_{k\in Z} P_k^{(i)}\Psi_l(ax-k), \tag{2.16}$$

$$\tilde{\Psi}_{al+i}(x) = \sum_{k\in Z}\tilde{P}_k^{(i)}\tilde{\Psi}_l(ax-k). \tag{2.17}$$

Properties and advantages of the biorthogonal multiwavelet packets with scale a are examined in Section 3. By applying the same method, one can obtain the orthogonal multiwavelet packets with scale a similar to those given by (2.16).

3. THE PROPERTIES OF BIORTHOGONAL MULTIWAVELET PACKETS WITH ARBITRARY SCALE

The main results of biorthogonality of the multiwavelet packets with scale a are presented in this section. Theorem 3.1 [12] and Lemmas 3.1 [4] and 3.2 [4] are listed below and used to prove the main results.

THEOREM 3.1. *Suppose*

$$\begin{aligned} W_j^{(i)} &= \mathrm{clos}_{L^2(R)} \langle \psi_{i_l;j,k} : 1 \le l \le t,\ k \in Z \rangle, \\ \tilde{W}_j^{(i)} &= \mathrm{clos}_{L^2(R)} \left\langle \tilde{\psi}_{i_l;j,k} : 1 \le l \le t,\ k \in Z \right\rangle, \qquad i = 1, 2, \ldots, a-1, \end{aligned} \tag{3.1}$$

then

$$W_j = \oplus_{i=1}^{a-1} W_j^{(i)}, \qquad \tilde{W}_j = \oplus_{i=1}^{a-1} \tilde{W}_j^{(i)}, \tag{3.2}$$

$$L^2(R) = \oplus_{j\in Z} W_j = \oplus_{j\in Z} \left(\oplus_{i=1}^{a-1} W_j^{(i)}\right) = \oplus_{j\in Z} \tilde{W}_j = \oplus_{j\in Z} \left(\oplus_{i=1}^{a-1} \tilde{W}_j^{(i)}\right), \tag{3.3}$$

where "$\oplus$" denotes direct sum of spaces.

LEMMA 3.1. *Suppose $\eta(x) = (\eta_1, \eta_2, \ldots, \eta_t)^\top$, $\tilde{\eta}(x) = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_t)^\top$, where $\eta_1, \eta_2, \ldots, \eta_t$, $\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_t \in L^2(R)$. Then the families $\{\eta_l(x-k) : 1 \le l \le t, k \in Z\}$ and $\{\tilde{\eta}_l(x-k) : 1 \le l \le t, k \in Z\}$ are biorthogonal function families if and only if*

$$\sum_{k\in Z} \hat{\eta}(\omega + 2k\pi)\hat{\tilde{\eta}}(\omega + 2k\pi)^* = I_t. \tag{3.4}$$

LEMMA 3.2. *Suppose Φ and $\tilde{\Phi}$ are biorthogonal scaling function vectors. $P(z)$ and $\tilde{P}(z)$ are their matrix symbols, respectively, ω_j, $j = 1, 2, \ldots, a$ are a roots of equation $z^a - 1 = 0$, then*

$$\sum_{j=1}^{a} P(\omega_j z)\tilde{P}(\omega_j z)^* = I_t, \qquad |z| = 1.$$

It is equal to

$$\sum_{i\in Z} P_i \tilde{P}_{i+ak} = a\delta_{0,k} I_t.$$

Suppose Ψ and $\tilde{\Psi}$ are biorthogonal multiwavelets with scale a, $Q(z)$, and $\tilde{Q}(z)$ are their matrix symbols, respectively, then

$$\sum_{j=1}^{a} P(\omega_j z)\tilde{Q}(\omega_j z)^* = 0, \qquad \sum_{j=1}^{a} \tilde{P}(\omega_j z)Q(\omega_j z)^* = 0,$$

$$\sum_{j=1}^{a} Q(\omega_j z)\tilde{Q}(\omega_j z)^* = I_{(a-1)t}.$$

These results are equivalent to the results below.

$$\sum_{i\in Z} P_i \tilde{Q}_{i+ak}^* = 0, \qquad \sum_{i\in Z} \tilde{P}_i Q_{i+ak}^* = 0, \qquad \sum_{i\in Z} Q_i \tilde{Q}_{i+ak}^* = a\delta_{0,k} I_{(a-1)t}.$$

One can easily prove Theorem 3.2 by applying the two lemmas.

THEOREM 3.2. *$P^{(i)}(z)$, $\tilde{P}^{(i)}(z)$ and $P_k^{(i)}$, $\tilde{P}_k^{(i)}$ are defined by (2.14),(2.15). If Ψ, $\tilde{\Psi}$ are the dual biorthogonal multiwavelets associated with Φ, $\tilde{\Phi}$, then*

$$\sum_{j=1}^{a} P^{(i)}(\omega_j z)\tilde{P}^{(m)}(\omega_j z)^* = 0, \qquad \sum_{j=1}^{a} \tilde{P}^{(i)}(\omega_j z)P^{(m)}(\omega_j z)^* = 0,$$
$$\sum_{j=1}^{a} P^{(i)}(\omega_j z)\tilde{P}^{(i)}(\omega_j z)^* = I_t, \tag{3.5}$$

which are equivalent to the results below,

$$\sum_{j\in Z} P_j^{(i)}\tilde{P}_{j+ak}^{(m)} = 0, \qquad \sum_{j\in Z} \tilde{P}_j^{(i)}P_{j+ak}^{(m)*} = 0, \qquad \sum_{j\in Z} P_j^{(i)}\tilde{P}_{j+ak}^{(i)*} = a\delta_{0,k}I_t, \tag{3.6}$$

where $i, m = 0, 1, \dots, a-1$, $i \neq m$.

THEOREM 3.3. *Suppose $n \in Z$ and*

$$n = \sum_{j=1}^{\infty} \varepsilon_j a^{j-1}, \qquad \varepsilon_j \in \{0, 1, 2, \dots, a-1\}, \tag{3.7}$$

then

$$\hat{\Psi}_n(\omega) = \prod_{j=1}^{\infty} P^{(\varepsilon_j)}\left(e^{-i\omega/a^j}\right)\hat{\Psi}_0(0), \tag{3.8}$$

$$\hat{\tilde{\Psi}}_n(\omega) = \prod_{j=1}^{\infty} \tilde{P}^{(\varepsilon_j)}\left(e^{-i\omega/a^j}\right)\hat{\tilde{\Psi}}_0(0). \tag{3.9}$$

PROOF. When $n = 0$, (3.8) holds. Suppose (3.8) holds for $0 \le n < a^{r_0}$. When $a^{r_0} \le n < a^{r_0+1}$, by introducing Fourier transforms on both sides of (2.16), and using inductive assumption and (3.7), we have

$$\begin{aligned}\hat{\Psi}_n(\omega) &= P^{(\varepsilon_1)}\left(e^{-i\omega/a}\right)\hat{\Psi}_{[n/a]}\left(\frac{\omega}{a}\right)\\ &= P^{(\varepsilon_1)}\left(e^{-i\omega/a}\right)\prod_{j=1}^{\infty} P^{(\varepsilon_{j+1})}\left(e^{-i\omega/a^{j+1}}\right)\hat{\Psi}_0(0) = \prod_{j=1}^{\infty} P^{(\varepsilon_j)}\left(e^{-i\omega/a^j}\right)\hat{\Psi}_0(0).\end{aligned}$$

Hence (3.8) holds by using induction. One can also obtain (3.9) by using the same argument. ∎

THEOREM 3.4. *For all $k, l \in Z$ and $n \in Z_+$,*

$$\left\langle \Psi_n(x-k), \tilde{\Psi}_n(x-l)\right\rangle = \delta_{k,l}I_t. \tag{3.10}$$

PROOF. When $n = 0$, (3.10) holds. Suppose (3.10) holds for $0 \le n < a^{r_0}$. When $a^{r_0} \le n < a^{r_0+1}$, by using inductive assumption and (3.4) and (3.5), we have

$$\begin{aligned}\left\langle \Psi_n(x-k), \tilde{\Psi}_n(x-l)\right\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Psi}_n(\omega)\hat{\tilde{\Psi}}_n(\omega)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\sum_{j\in Z}\int_{2a\pi j}^{2a\pi(j+1)} P^{(\varepsilon_1)}(z)\tilde{P}^{(\varepsilon_1)*}(z)\hat{\Psi}_{\lfloor n/a\rfloor}(\omega)\hat{\tilde{\Psi}}_{\lfloor n/a\rfloor}(\omega)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2a\pi} P^{(\varepsilon_1)}(z)\tilde{P}^{(\varepsilon_1)*}(z)\sum_{j\in Z}\hat{\Psi}_{\lfloor n/a\rfloor}(\omega+2a\pi j) \\ &\quad\cdot \hat{\tilde{\Psi}}_{\lfloor n/a\rfloor}(\omega+2a\pi j)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2a\pi} P^{(\varepsilon_1)}(z)\tilde{P}^{(\varepsilon_1)*}(z)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{m=0}^{a-1} P^{(\varepsilon_1)}(\omega_m z)\tilde{P}^{(\varepsilon_1)*}(\omega_m z)\right)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k-l)\omega} I_t\,\mathrm{d}\omega = \delta_{k,l}I_t.\end{aligned}$$

Hence (3.10) holds by induction. ∎

THEOREM 3.5. *For all $k, l \in Z$ and $n \in Z_+$ and $i \in \{1, 2, \ldots, a-1\}$, we have*

$$\left\langle \Psi_{an}(x-k), \tilde{\Psi}_{an+i}(x-l)\right\rangle = 0. \tag{3.11}$$

PROOF. By applying (2.16), (2.17), (3.4), and (3.5), we have

$$\begin{aligned}\left\langle \Psi_{an}(x-k), \tilde{\Psi}_{an+i}(x-l)\right\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P^{(0)}(z)\tilde{P}^{(i)*}(z)\hat{\Psi}_n(\omega)\hat{\tilde{\Psi}}_n(\omega)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2a\pi} P^{(0)}(z)\tilde{P}^{(i)*}(z)\sum_{m\in Z}\hat{\Psi}_n(\omega+2a\pi m) \\ &\quad\cdot \hat{\tilde{\Psi}}_n(\omega+2a\pi m)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2a\pi} P^{(\varepsilon_1)}(z)\tilde{P}^{(\varepsilon_1)*}(z)e^{-i(k-l)\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=1}^{a-1} P^{(0)}(\omega_j z)\tilde{P}^{(i)*}(\omega_j z)\right)e^{-i(k-l)\omega}\,\mathrm{d}\omega = 0.\end{aligned}$$

Hence (3.11) holds. ∎

THEOREM 3.6. *If $\{\Psi_n\}$, $\{\tilde{\Psi}_n\}$ are wavelet packets and dual wavelet packets, then for all $k \in Z$, we have decomposition formulae*

$$\Psi_n(ax-k) = \frac{1}{a^2}\sum_{j=1}^{a-1}\sum_{l\in Z}\tilde{P}^{(j)*}_{k-al}\Psi_{an+j}(x-l), \tag{3.12}$$

$$\tilde{\Psi}_n(ax-k) = \frac{1}{a^2}\sum_{j=1}^{a-1}\sum_{l\in Z}P^{(j)*}_{k-al}\tilde{\Psi}_{an+j}(x-l). \tag{3.13}$$

PROOF. By using (2.16), (3.6), we have

$$\begin{aligned}\frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\tilde{P}_{k-al}^{(j)*}\Psi_{an+j}(x-l) &= \frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\tilde{P}_{k-al}^{(j)*}\sum_{m\in Z}P_m^{(j)}\Psi_n(ax-al-m)\\ &= \frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\sum_{m\in Z}\tilde{P}_{k-al}^{(j)*}P_m^{(j)}\Psi_n(ax-al-m)\\ &= \frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{l\in Z}\sum_{r\in Z}\tilde{P}_{k-al}^{(j)*}P_{r-al}^{(j)}\Psi_n(ax-r)\\ &= \frac{1}{a^2}\sum_{j=0}^{a-1}\sum_{r\in Z}\Psi_n(ax-r)\sum_{l\in Z}\tilde{P}_{k-al}^{(j)*}P_{r-al}^{(j)}\\ &= \Psi_n(ax-k).\end{aligned}$$

Hence one can obtain (3.12). Similarly one can obtain (3.13) by using the same method. ∎

Equations (3.12) and (3.13) are used for performing decomposition. By using the same method, one can obtain a related formula for the orthogonal case as follows.

$$\Psi_n(ax-k) = \frac{1}{a^2}\sum_{j=1}^{a-1}\sum_{l\in Z}P_{k-al}^{(j)*}\Psi_{an+j}(x-l). \tag{3.14}$$

Note that the related cases of scalar wavelet packets have been proved in [15], but our proof proceeds in an easier way. By applying the above results, one can obtain the following result.

THEOREM 3.7. *For all $m, n = 0, 1, \dots,$ and $k, l \in Z$, we have*

$$\left\langle \Psi_m(x-k), \tilde{\Psi}_n(x-l)\right\rangle = \delta_{m,n}\delta_{k,l}I_t. \tag{3.15}$$

Finally note that the related results of orthogonal cases can be derived in a similar way.

4. THE DECOMPOSITION OF $L^2(R)$

We decompose subspaces V_j, $\tilde{V}_j$, W_j, $\tilde{W}_j$ further with biorthogonal multiwavelet packets with scale a and give the biorthogonal bases of $L^2(R)$ in this section. For convenience, we first define

$$\begin{aligned} U_n &= \operatorname{clos}_{L^2(R)}\langle \psi_{n_i}(x-k) : i = 1, 2, \dots, t,\ k \in Z\rangle,\\ \tilde{U}_n &= \operatorname{clos}_{L^2(R)}\left\langle \tilde{\psi}_{n_i}(x-k) : i = 1, 2, \dots, t,\ k \in Z\right\rangle,\end{aligned} \tag{4.1}$$

where ψ_{n_i} and $\tilde{\psi}_{n_i}$ are the i^{th} elements of Ψ_n and $\tilde{\Psi}_n$, respectively.

Define the scale operator δ as

$$\delta f := f(ax). \tag{4.2}$$

By applying (2.16), (2.17), and Theorem 3.6, we have

$$\begin{aligned} \delta^l U_n &= U_{a^l n} \oplus U_{a^l n+1} \oplus \cdots \oplus U_{a^l n + a^l - 1},\\ \delta^l \tilde{U}_n &= \tilde{U}_{a^l n} \oplus \tilde{U}_{a^l n+1} \oplus \cdots \oplus \tilde{U}_{a^l n + a^l - 1},\end{aligned} \tag{4.3}$$

where $l, n \in Z_+$, $\delta^0 = I$, and $\delta^l = \delta\cdot\delta^{l-1}$. Therefore it is possible to say that Theorem 3.5 is equivalent to the result,

$$U_{an+i}\perp\tilde{U}_{an+m}, \qquad i, m = 0, 1, \dots, a-1, \quad i \neq m.$$

THEOREM 4.1. *For $j = 0, 1, \dots,$*

$$V_j = \oplus_{0 \le n < a^j} U_n \qquad \text{and} \qquad W_j = \oplus_{a^j \le n < a^{j+1}} U_n, \tag{4.4}$$

$$\tilde{V}_j = \oplus_{0 \le n < a^j} \tilde{U}_n \qquad \text{and} \qquad \tilde{W}_j = \oplus_{a^j \le n < a^{j+1}} \tilde{U}_n, \tag{4.5}$$

where "$\oplus$" denotes direct sum of subspaces. $L^2(R)$ takes the form

$$L^2(R) = \oplus_{n \in Z} U_n = \oplus_{n \in Z} \tilde{U}_n, \tag{4.6}$$

where

$$U_m \perp \tilde{U}_n, \qquad m, n \in Z_+, \quad m \neq n, \tag{4.7}$$

and $\{\psi_{m_i}(x-k) : i = 1, 2, \dots, t,\ k \in Z\}$ and $\{\tilde{\psi}_{n_i}(x-k) : i = 1, 2, \dots, t,\ k \in Z\}$ are biorthogonal bases of U_m and $\tilde{U}_n$, respectively, where $m, n \in Z_+$.

PROOF. Note that $V_0 = U_0$, $W_0 = \oplus_{i=1}^{a-1} U_i$, $V_j = \delta^j V_0$, $W_j = \delta^j W_0$ and by using (4.3), one obtains (4.4) for all $j \in Z_+$. Similarly one obtains (4.5). Since $\{V_j\}_{j \in Z}$ and $\{\tilde{V}_j\}_{j \in Z}$ are all MRAs of $L^2(R)$, one obtains

$$L^2(R) = V_0 \oplus (\oplus_{j \in Z} W_j) = \tilde{V}_0 \oplus \left(\oplus_{j \in Z} \tilde{W}_j\right).$$

By using (4.4), (4.5), and Theorem 3.7, we obtain (4.6) and (4.7). ∎

THEOREM 4.2. *For all $l \in Z_+$, the families of function vectors*

$$\{\psi_{n_i;j,k} : a^l \le n < a^{l+1},\ i = 1, 2, \dots, t,\ j, k \in Z\} \tag{4.8}$$

and

$$\left\{\tilde{\psi}_{n_i;j,k} : a^l \le n < a^{l+1},\ i = 1, 2, \dots, t,\ j, k \in Z\right\} \tag{4.9}$$

are biorthogonal bases of $L^2(R)$ with the following meaning:

$$\left\langle \Psi_{n;j,k}, \tilde{\Psi}_{n';j',k'} \right\rangle = \delta_{n,n'} \delta_{j,j'} \delta_{k,k'} I_t, \tag{4.10}$$

where ψ_{n_i}, $\tilde{\psi}_{n_i}$ are the i^{th} elements of Ψ_n, $\tilde{\Psi}_n$. respectively.

PROOF. According to Theorems 3.4 and 4.1, $\{\psi_{n_i}(x-k) : a^l \le n < a^{l+1}, i = 1, 2, \dots, t,\ k \in Z\}$ and $\{\tilde{\psi}_{n_i}(x-k) : a^l \le n < a^{l+1}, i = 1, 2, \dots, t,\ k \in Z\}$ are biorthogonal bases of W_l and $\tilde{W}_l$. By using (4.1) and (4.2), one can easily show that $\{\psi_{n_i;j,k} : a^l \le n < a^{l+1},\ i = 1, 2, \dots, t,\ k \in Z\}$ and $\{\tilde{\psi}_{n_i;j,k} : a^l \le n < a^{l+1}, i = 1, 2, \dots, t,\ k \in Z\}$ are bases of $W_{l+j} = \delta^j W_l$ and $\tilde{W}_{l+j} = \delta^j \tilde{W}_l$, respectively. Since

$$L^2(R) = \oplus_{j \in Z} W_{l+j} = \oplus_{j \in Z} \tilde{W}_{l+j},$$

hence (4.8) and (4.9) are bases of $L^2(R)$. Noting that

$$V_{j+1} = V_j \oplus W_j, \qquad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j,$$

and

$$L^2(R) = \operatorname{clos}_{L^2(R)} \left\langle \bigcup_{j \in Z} V_j \right\rangle = \operatorname{clos}_{L^2(R)} \left\langle \bigcup_{j \in Z} \tilde{V}_j \right\rangle,$$

we have

$$W_{l+j} \perp \tilde{W}_{l+j'}, \qquad j, j' \in Z, \quad j \neq j'.$$

Hence (4.10) can be easily shown for the case when $j \neq j'$ and may be derived from Theorem 3.4 for the case $j = j'$. ∎

THEOREM 4.3.

$$\left\{\varphi_i(x-k), \psi_{n_i;(a-1)l(n)+j,k} : n,k \in Z,\ i=1,2,\dots,t,\ j=0,1,\dots,a-1,\ n \geq 1\right\} \tag{4.11}$$

and

$$\left\{\tilde{\varphi}_i(x-k), \tilde{\psi}_{n_i;(a-1)l(n)+j,k} : n,k \in Z,\ i=1,2,\dots,t,\ j=0,1,\dots,a-1,\ n \geq 1\right\} \tag{4.12}$$

are biorthogonal bases of $L^2(R)$ with the meaning as given by (4.10). Here

$$l(n) := \lfloor \log_a n \rfloor .$$

PROOF. For all $l \in Z_+$, considering $a^l \leq n < a^{l+1}$, by using the proof of Theorem 4.2, one can show that $\{\psi_{n_i;(a-1)l+j,k} : k \in Z,\ i=1,2,\dots,t,\ a^l \leq n < a^{l+1}\}$ is a basis of W_{al+j}, where $j=0,1,\dots,a-1$. Since

$$L^2(R) = V_0 \oplus (\oplus_{l\in Z} W_l) = U_0 \oplus \left(\oplus_{j=0}^{a-1} (\oplus_{l\in Z} W_{al+j})\right),$$

hence (4.11) is basis of $L^2(R)$. Similarly, one can show that (4.12) is a basis of $L^2(R)$. By using the proof of Theorem 4.2, one can show that (4.11) and (4.12) satisfy (4.10). ∎

We introduce two pairs of biorthogonal bases of $L^2(R)$ in the above two theorems.

5. ALGORITHMS OF DECOMPOSITION AND RECONSTRUCTION

Given a level N and consider

$$f \approx f_N := \sum_{j\in Z} C_j^N \Psi_0\left(a^N x - j\right) \in V_N,$$

where $\{C_j^N\}$ is a sequence consisting of t-dimensional constant vectors. By using the fact

$$V_N = W_{N-1} \oplus V_{N-1} = \cdots = W_{N-1} \oplus W_{N-2} \oplus \cdots \oplus W_{N-M} \oplus V_{N-M},$$

one obtains

$$f_N = g_{N-1} + g_{N-2} + \cdots + g_{N-M} + f_{N-M},$$

where $f_{N-M} \in V_{N-M}$ and $g_j \in W_j$, $j = N-M,\dots,N-1$.

Furthermore, by using Theorem 3.6, $g_j \in W_j$, $j=N-M,\dots,N-1$ can be further decomposed.

Suppose

$$f_j(x) = \sum_{k\in Z} C_k^j \Psi_0\left(a^j x - k\right), \qquad g_j(x) = \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j} \Psi_i\left(a^j x - k\right), \tag{5.1}$$

where $\{C_k^j\}_{k\in Z}$, $\{D_k^{i,j}\}_{k\in Z}$, $i=1,2,\dots,a-1$, $j=N-M,\dots,N-1$ are t-dimensional constant vectors. When $n=0$, by using (3.12), $f_j(x)$ can be decomposed as shown below.

$$\begin{aligned}
f_j(x) &= \sum_{k\in Z} C_k^j \Psi_0\left(a^j x - k\right) = \frac{1}{a^2} \sum_{k\in Z} C_k^j \sum_{i=0}^{a-1} \sum_{l\in Z} \tilde{P}_{k-al}^{(i)*} \Psi_{an+i}\left(a^j x - l\right) \\
&= \frac{1}{a^2} \sum_{k\in Z} \left(\sum_{l\in Z} C_l^j \tilde{P}_{l-ak}^{(0)*}\right) \Psi_0\left(a^{j-1}x - k\right) + \frac{1}{a^2} \sum_{i=1}^{a-1} \sum_{k\in Z} \left(\sum_{l\in Z} C_l^j \tilde{P}_{l-ak}^{(i)*}\right) \Psi_i\left(a^{j-1}x - k\right) \\
&= \sum_{k\in Z} C_k^{j-1} \Psi_0\left(a^{j-1}x - k\right) + \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j-1} \Psi_i\left(a^{j-1}x - k\right) \\
&= f_{j-1}(x) + g_{j-1}(x),
\end{aligned}$$

where

$$C_k^{j-1} = \frac{1}{a^2}\sum_{l\in Z} C_l^j \tilde{P}_{l-ak}^{(0)*}, \qquad D_k^{i,j-1} = \frac{1}{a^2}\sum_{l\in Z} C_l^j \tilde{P}_{l-ak}^{(i)*},$$
$$k \in Z, \qquad j = N, N-1, \ldots, N-M+1. \tag{5.2}$$

Using biorthogonal multiwavelet packets with scale a it is possible to further decompose $g_j(x)$. For all $r \in Z_+$,

$$g_j \in W_j = \delta^j W_1 = \delta^{j-r}\delta^r W_1 = \delta^{j-r} \oplus_{a^r \le i < a^{r+1}} U_i.$$

Using (3.12) when $n = 1, 2, \ldots, a^{r-1} - 1$, one obtains

$$g_j(x) = \sum_{i=1}^{a-1}\sum_{k\in Z} D_k^{i,j}\Psi_i\left(a^j x - k\right)$$
$$= \frac{1}{a^2}\sum_{i=1}^{a-1}\sum_{k\in Z} D_k^{i,j}\sum_{m=0}^{a-1}\sum_{l\in Z}\tilde{P}_{k-al}^{(m)*}\Psi_{ai+m}\left(a^{j-1}x - l\right)$$
$$= \frac{1}{a^2}\sum_{i=a}^{a^2-1}\sum_{k\in Z}\left(\sum_{l\in Z} D_l^{\lfloor i/a\rfloor, j}\tilde{P}_{l-ak}^{(i-a\lfloor i/a\rfloor)*}\right)\Psi_i\left(a^{j-1}x - k\right)$$
$$= \sum_{i=a}^{a^2-1}\sum_{k\in Z} D_k^{i,j,1}\Psi_i\left(a^{j-1}x - k\right)$$
$$\vdots$$
$$= \sum_{i=a^r}^{a^{r+1}-1}\sum_{k\in Z} D_k^{i,j,r}\Psi_i\left(a^{j-r}x - k\right),$$

where

$$D_k^{i,j,h} = \frac{1}{a^2}\sum_{l\in Z} D_l^{\lfloor i/a\rfloor, j, h-1}\tilde{P}_{l-ak}^{(i-a\lfloor i/a\rfloor)*}, \qquad D_k^{i,j,0} = D_k^{i,j},$$
$$h = 1, 2, \ldots, r, \qquad i = a^h, a^h + 1, \ldots, a^{h+1} - 1. \tag{5.3}$$

Hence, for $r \in Z_+$, f_N is decomposed as follows.

$$f_N = f_{N-M} + \sum_{j=N-M}^{N-1} g_j = \sum_{k\in Z} C_k^{N-M}\Psi_0\left(a^{N-M}x - k\right) + \sum_{j=N-M}^{N-1}\sum_{i=1}^{a-1}\sum_{k\in Z} D_k^{i,j}\Psi_i\left(a^j x - k\right)$$
$$= \sum_{k\in Z} C_k^{N-M}\Psi_0\left(a^{N-M}x - k\right) + \sum_{j=N-M}^{N-1}\sum_{i=a^r}^{a^{r+1}-1}\sum_{k\in Z} D_k^{i,j,r}\Psi_i\left(a^{j-r}x - k\right),$$

where $\{C_k^{N-M}\}$, $\{D_k^{i,j}\}$, $\{D_k^{i,j,r}\}$ are given by (5.2) and (5.3).

On the other hand, by using (2.16), $g_j(x)$ can be reconstructed as

$$g_j(x) = \sum_{i=a^m}^{a^{m+1}-1}\sum_{k\in Z} D_k^{i,j,r}\Psi_i\left(a^{j-r}x - k\right)$$
$$= \sum_{i=a^r}^{a^{r+1}-1}\sum_{k\in Z} D_k^{i,j,r}\sum_{l\in Z} P_l^{(i-a\lfloor i/a\rfloor)}\Psi_{\lfloor i/a\rfloor}\left(a^{j-r+1}x - ak - l\right)$$

$$= \sum_{i=a^{r-1}}^{a^r-1} \sum_{k\in Z} D_k^{i,j,r-1} \Psi_i \left(a^{j-r+1}x - k\right)$$

$$= \cdots$$

$$= \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j} \Psi_i \left(a^j x - k\right),$$

where

$$D_k^{i,j,h-1} = \sum_{n=0}^{a-1} \sum_{l\in Z} D_l^{ai+n,j,h} P_{k-al}^{(n)}, \qquad D_k^{i,j} = D_k^{i,j,0}, \tag{5.4}$$
$$h = 1,2,\dots,r, \qquad i = a^{h-1}, a^{h-1}+1, \dots, a^h - 1.$$

After obtaining $D_k^{i,j}$, $i = 1,2,\dots,a-1$, $j = N-M,\dots,N-1$, $k \in Z$, one can use (2.1) and (2.16) and derive the result below.

$$f_j = f_{j-1} + g_{j-1} = \sum_{k\in Z} C_k^{j-1} \Psi_0 \left(a^{j-1}x - k\right) + \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j-1} \Psi_i \left(a^{j-1}x - k\right)$$

$$= \sum_{k\in Z} C_k^{j-1} \sum_{l\in Z} P_l^{(0)} \Psi_0 \left(a^j x - ak - l\right) + \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j-1} \sum_{l\in Z} P_l^{(i)} \Psi_0 \left(a^j x - ak - l\right)$$

$$= \sum_{k\in Z} C_k^{j-1} \sum_{l\in Z} P_{l-ak}^{(0)} \Psi_0 \left(a^j x - l\right) + \sum_{i=1}^{a-1} \sum_{k\in Z} D_k^{i,j-1} \sum_{l\in Z} P_{l-ak}^{(i)} \Psi_0 \left(a^j x - l\right)$$

$$= \sum_{k\in Z} \left(\sum_{l\in Z} C_l^{j-1} P_{k-al}^{(0)} + \sum_{i=1}^{a-1} \sum_{l\in Z} D_l^{i,j-1} P_{k-al}^{(i)} \right) \Psi_0 \left(a^j x - k\right)$$

$$= \sum_{k\in Z} C_k^j \Psi_0 \left(a^j x - k\right),$$

where

$$C_k^j = \sum_{l\in Z} C_l^{j-1} P_{k-al}^{(0)} + \sum_{i=1}^{a-1} \sum_{l\in Z} D_l^{i,j-1} P_{k-al}^{(i)}, \qquad j = N-M+1, N-M+2, \dots, N, \quad k \in Z. \tag{5.5}$$

Hence, with given constant vector sequences $\{C_k^{N-M}\}_{k\in Z}$ $\{D_k^{i,N-M}\}_{k\in Z}$, $i = 1,2,\dots,a-1$, one can reconstruct $f \approx f_N = \sum_{k\in Z} C_k^N \Psi_0(a^N x - j) \in V_N$ by applying (5.5).

Similarly, the formulae for decomposition and reconstruction with orthogonal multiwavelet packets with scale a may be obtained by using the method below.

$$C_k^{j-1} = \frac{1}{a^2} \sum_{l\in Z} C_l^j P_{l-ak}^{(0)*}, \qquad D_k^{i,j-1} = \frac{1}{a^2} \sum_{l\in Z} C_l^j P_{l-ak}^{(i)*},$$
$$k \in Z, \qquad j = N, N-1, \dots, N-M+1,$$
$$D_k^{i,j,h} = \frac{1}{a^2} \sum_{l\in Z} D_l^{\lfloor i/a \rfloor, j, h-1} P_{l-ak}^{(i-a\lfloor i/a \rfloor)*}, \qquad D_k^{i,j,0} = D_k^{i,j}, \tag{5.6}$$
$$h = 1,2,\dots,r, \qquad i = a^h, a^h+1, \dots, a^{h+1}-1,$$

$$\begin{aligned} &D_k^{i,j,h-1} = \sum_{n=0}^{a-1} \sum_{l \in Z} D_l^{ai+n,j,h} P_{k-al}^{(n)}, \qquad D_k^{i,j,0} = D_k^{i,j}, \\ &h = 1, 2, \ldots, r, \qquad i = a^{h-1}, a^{h-1}+1, \ldots, a^h - 1, \\ &C_k^j = \sum_{l \in Z} C_l^{j-1} P_{k-al}^{(0)} + \sum_{i=1}^{a-1} \sum_{l \in Z} D_l^{i,j-1} P_{k-al}^{(i)}, \\ &j = N - M + 1, N - M + 2, \ldots, N, \qquad k \in Z. \end{aligned} \tag{5.6(cont.)}$$

The main steps of the algorithm for the decomposition described above can be summarized as follows. Given the signal f and the level N which satisfy

$$f \approx f_N := \sum_{j=L_1}^{L_2} C_j^N \Psi_0 \left(a^N x - j\right) \in V_N,$$

f may be decomposed by using biorthogonal multiwavelet packets with scale a. The algorithm is given below.

INPUT. $a, r, N, M, L_1, L_2, L_3, L_4 \in Z_+$, t-dimensional constant vector sequence $\{C_k^N\}$, $L_1 \leq k \leq L_2$ and $t \times t$ matrix sequence $\{\tilde{P}_k^{(i)}\}$, $i = 0, 1, 2, \ldots, a-1$, $L_3 \leq k \leq L_4$.

OUTPUT. t-dimensional constant vectors

$$\{C_k^{N-M}\}, \quad \{D_k^{i,j,r}\}, \quad L_1 \leq k \leq L_2, \quad a^r \leq i \leq a^{r+1} - 1, \quad N - M \leq j \leq N - 1.$$

STEP 1. $\{C_k^{N-M}\}$ and $\{D_k^{i,j}\}$, $L_1 \leq k \leq L_2$, $1 \leq i \leq a-1$, $N - M \leq j \leq N-1$, are given by (5.2).

STEP 2. $\{D_k^{i,j,r}\}$, $L_1 \leq k \leq L_2$, $a^r \leq i \leq a^{r+1} - 1$, $N - M \leq j \leq N - 1$, is given by (5.3).

The main steps of the algorithm for the reconstruction of the signal f are listed in the algorithm below.

INPUT. $a, r, N, M, L_1, L_2, L_3, L_4 \in Z_+$, t-dimensional constant vector sequences $\{C_k^{N-M}\}$, $\{D_k^{i,j,r}\}$, $L_1 \leq k \leq L_2$, $a^r \leq i \leq a^{r+1} - 1$, $N - M \leq j \leq N - 1$, and $t \times t$ matrix sequence $\{P_k^{(i)}\}$, $i = 0, 1, 2, \ldots, a-1$, $L_3 \leq k \leq L_4$.

OUTPUT. f.

STEP 1. $\{D_k^{i,j}\}$, $L_1 \leq k \leq L_2$, $1 \leq i \leq a - 1$, $N - M \leq j \leq N - 1$ is given by (5.4).

STEP 2. $\{C_k^N\}$, $L_1 \leq k \leq L_2$ is given by (5.5).

STEP 3. Calculate $f \approx f_N = \sum_{k \in Z} C_k^N \Psi_0(a^N x - j) \in V_N$.

As it is well known, signals are functions of time like music and speech at a given level. While stationary or quasi-stationary signals are decomposed into a series of time-frequency atoms using multiwavelets with scale a, transient signals can be analyzed. However typical information may be lost when small coefficients are set equal to zero and the other coefficients are quantized. In fact, signals, especially signals with high frequency, can be further decomposed by using the above algorithm for the decomposition, and analyzed. One can also reconstruct certain arriving signals by using the above algorithm for the reconstruction. Most of the multiplication process involved in the above algorithms may be decoupled and done in parallel. Such intrinsic parallel property is particularly useful in the actual implementation in parallel computers. However the summation process involved in the above algorithms requires some communication between computational nodes.

6. SUMMARY

We give a construction of multiwavelet packets with scale a in this paper. The properties of these packets are studied. The formulae for performing iterations and decomposition are given. We also derive two new bases of $L^2(R)$. This work provides a methodology of observing signals with high frequency. The algorithms for the decomposition and reconstruction using these packets are described and their implementation presented. Comments on parallel implementation are also given. The proposed algorithms may not be computationally simpler than those of [15], but are more flexible because the scale may be chosen arbitrarily.

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